JOURNAL OF COMBINATORIAL THEORY (A) 20, 398–403 (1976)

Note

Characterizing k-Flats in Geometric Designs

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Communicated by R. C. Bose

Received January 8, 1975

Let V be an n-dimensional vector space over $GF(q)$, and let U be a k-dimensional subspace of V. Let $[{}^{U}_{r}]$ denote the set of r-dimensional subspaces of U, and let $C(n, k, r, j) = \{|[{}^{W}_{r}] \cap [{}^{U}_{r}]|: W$ is an $(n-j)$-dimensional subspace of $V\}$. $C(n, k, r, j)$ is the set of cardinalities of "r-intersections" of U with all $(n-j)$-subspaces. In this paper we are concerned with the following question: Suppose S is a set of r-subspaces of V, and $|S| = [{}^{k}_{r}]$, the number of r-subspaces in a k-space over $GF(q)$. Then if $\{|S \cap [{}^{W}_{r}]|: W$ is an $(n-j)$-subspace of $V\} \subseteq C(n, k, r, j)$, must $S = [{}^{U}_{r}]$ for some k-subspace $U \subseteq V$? In [4, 5] the answer was shown to be affirmative if $r = 1$ or $j = 1$. In this paper we show that the answer is affirmative for arbitrary v and j provided that n is sufficiently large. The minimum sufficient size for n is not decided here. The bound we obtain is clearly too crude.

What we actually do here is to prove a theorem for somewhat more general objects than vector spaces. One special case is that of graphs. In this case we get a theorem related to the Ulam reconstruction conjecture. In particular, we can reconstruct the complete graphs from only the number of edges in the maximal subgraphs in certain cases. This also suggests the q-analog of the reconstruction conjecture.

* Work partially supported by NSF Grant GP-33580 and the Alfred P. Sloan Foundation.

** Work partially supported by NSF Grant 40583.

Let G be a combinatorial geometry with rank function α. We will follow Crapo and Rota [1] for terminology and notation. G will be called a *geometric design* if all k-flats (closed sets of rank k) have the same cardinality $\bar{k}$. We will have $\bar{0} = 0$, $\bar{1} = 1$. Geometric designs were first defined by Edmonds, Murti, and Young [2], who called them simple matroid designs. Many examples of geometric designs are given in [2], and we list four of them here.

(a) n-dimensional projective space over $GF(q)$. The k-flats are $(k-1)$-dimensional subspaces, and we have $\bar{k} = (q^k - 1)/(q - 1)$.

(b) n-dimensional affine space over $GF(q)$. The k-flats are $(k-1)$-dimensional subspaces, we have $\bar{k} = q^{k-1}$, $k \geqslant 1$, $\bar{0} = 0$.

(c) The subsets of an n element set. The k-flats are the sets of size k, and $\bar{k} = k$.

(d) The geometric design G formed from a $t-(v, k, \lambda)$ design (see [1]) as follows. A $t-(v, k, \lambda)$ design D [4] is a pair (X, L) where X is a set with $|X| = v$, L is a family of subsets of X such that for all $B \in L$, $|B| = k$, and for any t-subsets $Y \subseteq X$, there are exactly λ elements $A \in L$ such that $Y \subseteq A$.

We define a geometric design as follows: the k-flats are all the k-subsets of X if $k < t$, the numbers of L if $k = t$, and X itself if $k = \overline{t+1}$. Thus, $\bar{k} = k$ if $k \leqslant t$, $t + 1 = v = |X|$.

Now let G be a geometric design. We will denote by $\left[{S \atop r}\right]$ the set of r-flats contained in the set S. If S is a k-flat, then the usual argument of counting bases in vector spaces can be easily extended to show that $\left|\left[{S \atop r}\right]\right|$ is independent of the choice of k-flat, and that

$$\left[{k \atop r}\right] = \left|\left[{S \atop r}\right]\right| = \frac{\bar{k}(\bar{k} - \bar{1}) \cdots (\bar{k} - (\overline{r-1}))}{\bar{r}(\bar{r} - \bar{1}) \cdots (\bar{r} - (\overline{r-1}))} \qquad r \geqslant 1$$

[1]. Note that $r > k$ implies $\left[{k \atop r}\right] = 0$.

Let G be a geometric design of rank n. Then $C_G(n, k, r, j)$ (or just $C(n, k, r, j)$) will denote the set $\{0, \left[{k \atop r}\right], ..., \left[{k-j \atop r}\right]\}$. We say that a set R of r-flats of G satisfies $P(n, k, r, j)$ if

(i) $|R| = \left[{k \atop r}\right]$

(ii) $\left|R \cap \left[{A \atop r}\right]\right| \in C(n, k, r, j)$ for every $(n-j)$-flat A of G.

If G is any of examples (a), (b), or (c) above, and if $R = \left[{K \atop r}\right]$ for some k-flat K, then R satisfies $P(n, k, r, j)$.

We will say that R satisfies $Q(n, k, r, j)$ if (i) above holds together with

(iii) $\left|R \cap \left[{A \atop r}\right]\right| = \left[{k \atop r}\right]$ or $\left|R \cap \left[{A \atop r}\right]\right| \leqslant \left[{k-1 \atop r}\right]$ for every $(n-j)$-flat A of G.

Obviously, $P(n, k, r, j)$ implies $Q(n, k, r, j)$. In any geometric design G if $R = [{K \atop r}]$ for some k-flat K, then R satisfies $Q(n, k, r, j)$.

If for some geometric design G a set R of r-flats satisfies $Q(n, k, r, j)$ (respectively, $P(n, k, r, j)$) if and only if $R = [{K \atop r}]$ for some k-flat K, then we say that $Q(n, k, r, j)$ (respectively, $P(n, k, r, j)$) *characterizes* k-flats of G.

Rothschild and van Lint [5] proved in examples (a) and (b) above that $P(n, k, r, j)$ characterizes k-flats (except for $q = 2$, $r = 1$, $j = n - 2$ in case (b)) if $r = 1$ or $j = 1$, and $n - 2 \geqslant j$. In this note we will prove that if G is a geometric design of rank n satisfying the following condition

$$(*) \qquad \frac{\bar{k}}{\overline{k-1}} \leqslant \frac{\overline{k-1}}{\overline{k-2}}, \qquad 2 \leqslant k \leqslant n$$

then $Q(n, k, r, j)$ characterizes k-flats of G provided that $n \geqslant r[{k \atop r}] + j - 1$.

We note that (a), (b), and (c) above satisfy (*). But not all goemetric designs do. For example, some t-designs (example (d)) give rise to geometric designs not satisfying (*).

THEOREM 1. *Let G be a geometric design of* rank n *satisfying* (*). *Let* $n \geqslant r[{k \atop r}] + j - 1$. *Then $Q(n, k, r, j)$ characterizes k-flats of G.*

First we prove two lemmas (see also [2]).

LEMMA 1. *Let G be a geometric design of rank n. Let K be a k-flat of G, $0 \leqslant k \leqslant n$. Let $j \geqslant 0$, and $M = \{S \mid S$ is a j-flat of G with $K \leqslant S\}$. Then $|M|$ is independent of the choice of K and is given by* $|M| = ([{j \atop k}]/[{n \atop k}])[{n \atop j}] = t(n, k, j)$.

Proof. This formula follows by counting bases as above for the formula for $[{k \atop r}]$.

LEMMA 2. *Let G be a geometric design satisfying* (*). *Let $k_1 > k$. Then*

(i) $\dfrac{\bar{k}_1}{\overline{(k_1 - 1)}} \leqslant \dfrac{\bar{k}}{\overline{(k-1)}}$ *if* $k \geqslant 1$

(ii) $\bar{k}_1 - \overline{(k_1 - 1)} \geqslant \bar{k} - \overline{(k-1)}$

(iii) $\bar{k}_1 - \overline{(k_1 - 1)} > \bar{k} - \overline{(k-1)}$ *if* $\dfrac{\bar{k}_1}{\overline{(k_1 - 1)}} = \dfrac{\bar{k}}{\overline{(k-1)}}$

(iv) $\dfrac{\begin{bmatrix} k_1 \\ r \end{bmatrix}}{\begin{bmatrix} k_1 - 1 \\ r \end{bmatrix}} < \dfrac{\begin{bmatrix} k \\ r \end{bmatrix}}{\begin{bmatrix} k-1 \\ r \end{bmatrix}}$ *if* $k \geqslant r \geqslant 2$.

Proof. (i) and (iii) are obvious, and (ii) is easily proved using the exchange property of geometries. (iv) follows from (i) and the elementary fact that since $k_1 > k$, and thus, $\overline{(k_1 - 1)} > \overline{(k - 1)}$, etc., we always have $(\bar{k}_1 - c)/(\overline{(k_1 - 1)} - c) < (\bar{k} - c)/(\overline{(k - 1)} - c)$ for all $0 < c < \overline{(k - 1)}$.

Proof of Theorem 1. Let R be a set of r-flats of G satisfying $Q(n, k, r, j)$. Let $K_R = \{x \mid \exists l \in R \text{ with } x \in l\}$, and let $\bar{K}_R$ be the closure of K_R (the smallest flat containing it). Let the rank of $\bar{K}_R$ be $\alpha(\bar{K}_R) = k_1$. We note that $R \subseteq \genfrac{[}{]}{0pt}{}{\bar{K}_R}{r}$.

First, if $k_1 \leqslant k$, we have $\genfrac{[}{]}{0pt}{}{k}{r} = |R| \leqslant |\genfrac{[}{]}{0pt}{}{\bar{K}_R}{r}| = \genfrac{[}{]}{0pt}{}{k_1}{r} \leqslant \genfrac{[}{]}{0pt}{}{k}{r}$. Hence $k = k_1$ and $R = \genfrac{[}{]}{0pt}{}{\bar{K}_R}{r}$.

So assume $k_1 > k$. Let $\{e_1, ..., e_{k_1}\} \subseteq K_R$ be a basis for $\bar{K}_R$. Let $M = \{H \mid H \text{ is an } (n-j)\text{-flat of } G \text{ such that the rank } \alpha(H \cap \bar{K}_R) = k_1 - 1\}$. If S is any $(k_1 - 1)$-flat of $\bar{K}_R$, then S is contained in exactly $t(n, k_1 - 1, n - j)$ $(n - j)$-flats of G by Lemma 1. Exactly $t(n, k_1, n - j)$ of these $(n - j)$-flats will contain $\bar{K}_R$. Thus

$$|M| = (t(n, k_1 - 1, n - j) - t(n, k_1, n - j)) \begin{bmatrix} k_1 \\ k_1 - 1 \end{bmatrix}. \tag{1}$$

For any $H \in M$, let $d(H)$ denote the number of $l \in R$ such that $l \subseteq H$, i.e., $d(H) = |\genfrac{[}{]}{0pt}{}{H}{r} \cap R|$. Not all the e_i can be in H or else $H \cap \bar{K}_R = \bar{K}_R$. Thus $d(H) < \genfrac{[}{]}{0pt}{}{k}{r}$. By $Q(n, k, r, j)$ we have $d(H) \leqslant \genfrac{[}{]}{0pt}{}{k-1}{r}$ for all $H \in M$. Thus

$$\sum_{H \in M} d(H) \leqslant |M| \begin{bmatrix} k - 1 \\ r \end{bmatrix}. \tag{2}$$

By Lemma 1, any $l \in R$ is contained in exactly $t(k_1, r, k_1 - 1)$ $(k_1 - 1)$-flats of $\bar{K}$ and thus in exactly $t(k_1, r, k_1 - 1)(t(n, k_1 - 1, n - j) - t(n, k_1, n - j))$ elements of M. Thus

$$\sum_{H \in M} d(H) = t(k_1, r, k_1 - 1)(t(n, k_1 - 1, n - j) - t(n, k_1, n - j)) \begin{bmatrix} k \\ r \end{bmatrix}. \tag{3}$$

If we take a basis for each $l \in R$, their union certainly spans $\bar{K}_R$. Thus, $k_1 \leqslant r\genfrac{[}{]}{0pt}{}{k}{r}$ and by the assumption of the theorem, $n - j \geqslant k_1 - 1$. Hence $t(n, k_1 - 1, n - j) - t(n, k_1, n - j) > 0$. Thus we can combine (2) and (3) and use Lemma 1 to obtain:

$$\frac{\begin{bmatrix} k \\ r \end{bmatrix}}{\begin{bmatrix} k - 1 \\ r \end{bmatrix}} \leqslant \frac{\begin{bmatrix} k_1 \\ r \end{bmatrix}}{\begin{bmatrix} k_1 - 1 \\ r \end{bmatrix}}. \tag{4}$$

If $r \geqslant 2$, this contradicts Lemma 2(iv).

Thus we assume $r = 1$. If for any $H \in M$ we have $d(H) < \left[{k-1 \atop r}\right] = \overline{(k-1)}$, then the inequality in (2) is strict, and hence the inequality in (4) is strict, contradicting Lemma 2, (i). Thus we must have $d(H) = \overline{(k-1)}$ for all $H \in M$, and equality in (4).

We now proceed as in [5]. Let χ_H be the characteristic function of the set H. Then $\overline{(k-1)}^2 \mid M \mid = \sum_{H\in M} (d(H))^2 = \sum_{H\in M} (\sum_{l\in R} \chi_H(l))^2$. So $\overline{(k-1)}^2 \mid M \mid = \sum_{l,l'\in R} (\sum_{H\in M} \chi_H(l)\, \chi_H(l')) = \sum_{l\in R} \sum_{H\in M} (\chi_H(l))^2 + \sum_{l\neq l'\in R} \sum_{H\in M} \chi_H(l)\, \chi_H(l') = (\mid R \mid t(k_1, 1, k_1 - 1) + \mid R \mid (\mid R \mid - 1)\, t(k_1, 2, k_1 - 1)) \cdot (t(n, k_1 - 1, n - j) - t(n, k_1, n - j))$. Now using the formulas for $\mid M \mid$ from (1), for $t(n, k, j)$ and for $\left[{k \atop r}\right]$, we get:

$$\overline{(k-1)}^2 \begin{bmatrix} k_1 \\ k_1 - 1 \end{bmatrix} = \bar{k}\,\frac{\overline{(k_1-1)}}{\bar{k}_1} \begin{bmatrix} k_1 \\ k_1 - 1 \end{bmatrix} + \bar{k}(\bar{k}-1)\,\frac{\begin{bmatrix} k_1 - 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} k_1 \\ 2 \end{bmatrix}} \begin{bmatrix} k_1 \\ k_1 - 1 \end{bmatrix}$$

or

$$\overline{(k-1)}^2 = \frac{\bar{k}\,\overline{(k_1-1)}}{\bar{k}_1} + \frac{\bar{k}(\bar{k}-1)\,\overline{(k_1-1)}\,(\overline{(k_1-1)}-1)}{\bar{k}_1(\bar{k}_1-1)}.$$

Using the equality in (4) again, this gives $\bar{k} - \overline{(k-1)} = \bar{k}_1 - \overline{(k_1-1)}$, contradicting Lemma 2(iii), and completing the proof of Theorem 1.

We have shown that $Q(n, k, r, j)$ characterizes k-flats provided $n \geqslant r\left[{k \atop r}\right] + j - 1$. This fact was used only once in the proof above in order to guarantee that $k_1 - 1 \leqslant n - j$. k_1 was the rank of the smallest flat containing all f-flats in R. But it is obvious that in order for k_1 to be anywhere near the value $r\left[{k \atop r}\right]$, R would necessarily not satisfy $Q(n, k, r, j)$. Thus the bounds could be easily improved.

We will consider briefly the case of example (c), subsets of a set, and in particular the $r = 2$ case. This is the case of graphs. Namely, suppose we have a graph G on n vertices and $\binom{k}{2}$ edges. Suppose every induced subgraph on $n - j$ vertices has either $\binom{k}{2}$ edges or at most $\binom{k-1}{2}$ edges. Then if n is large enough, Theorem 1 guarantees that G consists of a complete graph on k vertices together with $n - k$ isolated vertices. If $f(k, j)$ is the smallest n for which this property holds (i.e., $Q(n, k, 2, j)$ implies that G has this structure), then Theorem 1 implies that $f(k, j) \leqslant O(k^2)$ for fixed j.

Even a rather crude argument can improve this to $O(k)$. For suppose $j - 1$ of the vertices are isolated. Then using these in the complement of an $n - j$ set we see that any vertex with positive degree has degree at least $k - 1$ by property Q. This forces G to be the desired graph. On the other hand, suppose at most $j - 1$ vertices have degree 0. Then if we take $n \geqslant (k + 1)j$, and divide the vertices into $(k + 1)$ disjoint j-sets, each

j-set must meet at least $(k - 1)$ edges of G, by property Q. This requires more than $\binom{k}{2}$ edges. Thus $f(k,j) \leqslant O(k)$. Since k vertices at least are required to have Q satisfied at all, we get $f(k,j) = O(k)$.

Similar arguments apply more generally. It may be true that for some c, $f(k,j) \leqslant k + c$.

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